

Torsion Effects in Braneworld Scenarios

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We present gravitational aspects of braneworld models endowed with torsion terms both in the bulk and on the brane. In order to investigate a conceivable and measurable gravitational effect, arising genuinely from bulk torsion terms, we analyze the variation in the black hole area by the presence of torsion. Furthermore, we extend the well known results about consistency conditions in a framework that incorporates brane torsion terms. It is shown, in a rough estimate, that the resulting effects are generally suppressed by the internal space volume. This formalism provides manageable models and their possible ramifications into some aspects of gravity in this context, and cognizable corrections and physical effects as well.

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I. INTRODUCTION

The presence of extra dimensions seems to be an almost inescapable characteristic of high-energy physics based upon the auspices of string theory. In this context, specific string theory inspired scenarios, in which our universe is modeled by a brane — the braneworld scenario — acquired special attention [1] due to the possibility of solving the hierarchy problem. Concomitantly, the presence of torsion is also an output of string theory [2]. Indeed, when gravitation is recovered from string theory, a plenty of physical fields abound, including the torsion field. In this context, among other motivations, it seems natural to explore some properties of braneworld models in the presence of torsion.

The easiest method of introducing torsion terms in the theory is via the addition of an antisymmetric part in the affine connection. The torsion tensor components can be written in terms of the connection components $\Gamma^\rho_{\beta\alpha}$ as $T^\rho_{\alpha\beta} = \Gamma^\rho_{\beta\alpha} - \Gamma^\rho_{\alpha\beta}$. The general connection components are related to the Levi-Civita connection components $\mathring{\Gamma}^\rho_{\alpha\beta}$ — associated with the spacetime metric $g_{\alpha\beta}$ components — through $\Gamma^\rho_{\alpha\beta} = \mathring{\Gamma}^\rho_{\alpha\beta} + K^\rho_{\alpha\beta}$, where $K^\rho_{\alpha\beta} = \frac{1}{2}(T_\alpha^\rho{}_\beta + T_\beta^\rho{}_\alpha - T^\rho{}_{\alpha\beta})$ denotes the contortion tensor components. Hereon the quantities denoted by \mathring{X} are constructed with the usual metric compatible torsionless Levi-Civita connection components $\mathring{\Gamma}^\rho_{\alpha\beta}$. We remark that the source of contortion may be considered as the rank-2 antisymmetric potential Kalb-Ramond (KR) field $B_{\alpha\beta}$, arising as a massless mode in heterotic string theories [2, 3]. Hereon we shall consider the formal geometric contortion, although the contortion induced by the KR field can be considered in the 5-dimensional formalism when the prescription $K^\rho_{\alpha\beta} = -\frac{1}{M^{3/2}}H^\rho_{\alpha\beta}$ is taken into account, where $H_{\rho\alpha\beta} = \partial_{[\rho}B_{\alpha\beta]}$ and M denotes the 5-dimensional Planck mass. The identification between the KR field and the contortion can be always taken into account when necessary, depending on the physical aspect of the formalism that must be emphasized, although the formalism is not precisely concerned with the fount of contortion, but with its consequences.

The combination of braneworld ideas encompassing torsion terms seems to be an interesting approach in trying to ascertain new signatures coming from high-energy physics. A complete scenario, however, lacks, and we propose to delve into such a framework. In this sense, some subterfuges are needed in order to extract physical information from the model dealt with. The main aim of this paper is to apply two well known strategies commonly used in braneworld scenarios, in the context of models involving torsion: the sum rules, bringing information about the brane torsion terms behavior, and the Taylor expansion outside a black hole metric, which gives information about the bulk torsion terms, where the corrections in the area of the 5D black string horizon are evinced. In order to find some typical gravitational signatures of braneworld scenarios with torsion we obtain all the formulæ for a Taylor expansion outside a black hole in Section II, extending some results of Ref. [4] in order to encompass accrued torsion corrections. It is shown how the contortion and its derivatives determine the variation in the area of the black hole horizon along the extra dimension, inducing observable physical effects.

In Section III we apply the braneworld consistency conditions in the case when torsion is present in the brane

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manifold. We are particularly concerned with the viability of such an extension, analyzing the torsion effects in the brane scalar curvature. We show that, for factorizable metrics, the torsion contribution to the brane curvature is damped by the distance between the branes. In warped braneworld models, however, this damping is — at least partially — compensated by terms of the warp factor. It is also shown that if the brane manifold is endowed with a connection presenting torsion then a Randall-Sundrum like scenario with equal sign brane tension becomes possible, in acute contrast to the standard Randall-Sundrum model. Roughly speaking, this last possibility comes from the following reasoning. The presence of the torsion terms generally relax the consistency conditions, specially in what concerns the sum over the brane tensions. Therefore, the brane tensions are not restricted to the same sign, although constrained by a specific contraction of contortion terms.

Our program throughout this paper explicitly consists of the following: the next Section deals with the question of finding a possible departure from the usual (torsionless) braneworld scenario, induced exclusively from bulk torsion terms within braneworld scenarios endowed with a general affine connection presenting torsion. Particularly, it is accomplished by looking the variation in the black hole horizon due to an specific arrangement of contortion terms. In Section III, we analyze some aspects related to the feasibility of braneworld manageable models with brane torsion terms. By studying the general consistency conditions applied to this case, we arrive at some roughly estimates concerning brane torsion effects. Although the presence of torsion is not prohibited at all, its effects are generally suppressed.

Our generalization concerns to construct a formal framework that contains terms of contortion, capable — for instance — to correct black holes horizons and consequent measurement of variation in quasars luminosities, as in [5] and references therein. Also, the braneworld consistency conditions are investigated in terms of contortion terms corrections demonstrating that for factorizable metrics, the contortion contribution to the brane curvature is damped by the distance between the branes.

II. MEASURABLE TORSION EFFECTS

In a previous paper [6] we proved that although the presence of torsion terms in the connection does not modify the Israel-Darmois matching conditions, despite of the modification in the extrinsic curvature and in the connection, the Einstein equation, obtained using the Gauss-Codazzi formalism, is extended. The factors involving contortion alter drastically the effective Einstein equation on the brane, and also the function involving contortion terms that is analogous to the effective cosmological constant as well.

We shall use such results to extend the bulk metric Taylor expansion in terms of the brane metric, in a direction orthogonal to the brane, encompassing torsion terms. As an immediate application, the corrections in a black hole horizon area due to contortion terms are achieved.

We have investigated the matching conditions in the presence of torsion terms, and under the assumptions of discontinuity across the brane, both of the junction conditions are shown to be the same as the usual case. As the covariant derivative is modified by the contortion, the extrinsic curvature is also modified, and the conventional arguments point in the direction of some modification in the matching conditions. However, it seems that the role of torsion terms in the braneworld picture is restricted to the geometric part of effective Einstein equation on the brane. More explicitly, looking at the equation that relates the Einstein equation in four dimensions with bulk quantities (see reference [7], for instance) it follows that

$${}^{(4)}G_{\rho\sigma} = \frac{2k_5^2}{3} \left(T_{\alpha\beta} q_\rho^\alpha q_\sigma^\beta + (T_{\alpha\beta} n^\alpha n^\beta - \frac{1}{4}T) q_{\rho\sigma} \right) + \Xi \Xi_{\rho\sigma} - \Xi_\rho^\alpha \Xi_{\alpha\sigma} - \frac{1}{2} q_{\rho\sigma} (\Xi^2 - \Xi^{\alpha\beta} \Xi_{\alpha\beta}) - {}^{(5)}C_{\beta\gamma\epsilon}^\alpha n_\alpha n^\gamma q_\rho^\beta q_\sigma^\epsilon, \quad (1)$$

where $T_{\rho\sigma}$ denotes the energy-momentum tensor, $\Xi_{\rho\sigma} = q_\rho^\alpha q_\sigma^\beta \nabla_\alpha n_\beta$ is the extrinsic curvature, k_5 denotes the 5-dimensional gravitational constant, and ${}^{(5)}C_{\beta\rho\sigma}^\alpha$ denotes the Weyl tensor. By restricting to quantities evaluated on the brane, or tending to the brane, we see that the only way to get some contribution from torsion terms is via the term ${}^{(4)}G_{\rho\sigma}$, and also via the Weyl tensor. Supposing \mathbb{Z}_2 -symmetry, the extrinsic curvature reads

$$\Xi_{\alpha\beta}^+ = -\Xi_{\alpha\beta}^- = -2G_N \left(\pi_{\alpha\beta} - \frac{q_{\alpha\beta}\pi_\gamma^\gamma}{4} \right). \quad (2)$$

Decomposing the stress-tensor associated with the bulk in $T_{\alpha\beta} = -\Lambda g_{\alpha\beta} + \delta S_{\alpha\beta}$ and $S_{\alpha\beta} = -\lambda q_{\alpha\beta} + \pi_{\alpha\beta}$, where Λ is the bulk cosmological constant and λ the brane tension, and substituting into Eq.(1) it follows that

$${}^{(4)}G_{\mu\nu} = -\Lambda_4 q_{\mu\nu} + 8\pi G_N \pi_{\mu\nu} + k_5^4 Y_{\mu\nu} - E_{\mu\nu}, \quad (3)$$

where $E_{\mu\nu} = {}^{(5)}C_{\beta\gamma\sigma}^\alpha n_\alpha n^\gamma q_\mu^\beta q_\nu^\sigma$ encodes the Weyl tensor contribution, $G_N = \frac{\lambda k_5^4}{48\pi}$ is the analogue of the Newton gravitational constant, the tensor $Y_{\mu\nu}$ is quadratic in the brane stress-tensor and given by $Y_{\mu\nu} = -\frac{1}{4}\pi_{\mu\alpha}\pi_\nu^\alpha + \frac{1}{12}\pi_\gamma^\gamma\pi_{\mu\nu} + \frac{1}{8}q_{\mu\nu}\pi_{\alpha\beta}\pi^{\alpha\beta} - \frac{1}{2}q_{\mu\nu}(\pi_\gamma^\gamma)^2$, and $\Lambda_4 = \frac{k_5^2}{2}\left(\Lambda + \frac{1}{6}k_5^2\lambda^2\right)$ denotes the effective brane cosmological constant.

Using the Einstein tensor on the brane encoding torsion terms, the $E_{\mu\nu}$ tensor can be expressed in terms of the bulk contortion terms by

$$\begin{aligned} E_{\kappa\delta} &= \mathring{E}_{\kappa\delta} + \left(\nabla_{[\nu}K_{\alpha\beta]}^\mu + K_{\gamma[\nu}^\mu K_{\alpha\beta]}^\gamma\right)n_\mu n^\nu q_\kappa^\alpha q_\delta^\beta - \frac{2}{3}(q_\kappa^\alpha q_\delta^\beta + n^\alpha n^\beta q_{\kappa\delta})\left(\nabla_{[\lambda}K_{\beta\alpha]}^\lambda + K_{\gamma\lambda}^\lambda K_{\beta\alpha}^\gamma - K_{\beta\gamma}^\sigma K_{\sigma\alpha}^\gamma\right) \\ &\quad + \frac{1}{6}q_{\kappa\delta}\left(2\nabla^\lambda K_{\lambda\tau}^\tau - K_{\tau\lambda}^\lambda K_{\tau\gamma}^\gamma + K_{\tau\gamma\lambda} K_{\tau\lambda\gamma}^\gamma\right), \end{aligned} \quad (4)$$

where ∇_μ is the bulk covariant derivative. Now, the explicit influence of the contortion terms in the Einstein brane equation can be visualized. From Eqs.(3), (4) and expressing the torsion terms of the Einstein brane tensor (see Eq. (20) of reference [6]), it follows that

$$\begin{aligned} {}^{(4)}\mathring{G}_{\mu\nu} + D_{[\lambda}{}^{(4)}K_{\mu\nu]}^\lambda + {}^{(4)}K_{\gamma\delta}^\delta {}^{(4)}K_{\mu\nu}^\lambda - {}^{(4)}K_{\nu\gamma}^\sigma {}^{(4)}K_{\sigma\mu}^\gamma = -\tilde{\Lambda}_4 q_{\mu\nu} + 8\pi G_N \pi_{\mu\nu} + k_5^4 Y_{\mu\nu} - \mathring{E}_{\mu\nu} \\ + q_\mu^\alpha q_\nu^\beta \left[\frac{2}{3}\left(\nabla_{[\lambda}K_{\beta\alpha]}^\lambda + K_{\gamma\sigma}^\sigma K_{\beta\alpha}^\gamma - K_{\beta\gamma}^\lambda K_{\lambda\alpha}^\gamma\right) - n_\rho n^\sigma \left(\nabla_{[\sigma}K_{\alpha\beta]}^\rho + K_{\gamma[\sigma}^\rho K_{\alpha\beta]}^\gamma\right) \right], \end{aligned} \quad (5)$$

where

$$\begin{aligned} \tilde{\Lambda}_4 &\equiv \Lambda_4 - D^\lambda {}^{(4)}K_{\lambda\tau}^\tau + \frac{1}{2} {}^{(4)}K_{\tau\alpha}^\alpha {}^{(4)}K_{\tau\lambda}^\lambda - \frac{1}{2} {}^{(4)}K_{\tau\gamma\lambda} {}^{(4)}K_{\tau\lambda\gamma}^\lambda - \frac{2}{3}n^\alpha n^\beta \left(\nabla_\lambda K_{\beta\alpha}^\lambda - \nabla_\alpha K_{\beta\lambda}^\lambda\right. \\ &\quad \left.+ K_{\gamma\lambda}^\lambda K_{\beta\alpha}^\gamma - K_{\beta\gamma}^\sigma K_{\sigma\alpha}^\gamma\right) + \frac{1}{6}\left(2\nabla^\lambda K_{\lambda\tau}^\tau - K_{\tau\alpha}^\alpha K_{\tau\lambda}^\lambda + K_{\tau\gamma\lambda} K_{\tau\lambda\gamma}^\gamma\right). \end{aligned} \quad (6)$$

The function $\tilde{\Lambda}_4$ above is usually called effective cosmological “constant” in the literature, in the sense that all its terms are multiplied by the brane metric in the Einstein effective equation (5). Eqs. (5) and (6) show that the factors involving contortion, both in four and in five dimensions, modify drastically the effective Einstein equation on the brane and the effective cosmological constant as well.

Now, let us look at some deviations of the black hole horizon coming from the bulk torsion terms. Hereon in this Section we assume vacuum on the brane ($\pi_{\mu\nu} = 0 = Y_{\mu\nu}$) and neglect the contribution of the effective cosmological constant term, which is expected to be smaller, by some orders of magnitude, than the contribution of the term Weyl [4]. Using a Taylor expansion in the extra dimension in order to probe properties of a static black hole on the brane [8], the bulk metric can be written as

$$\begin{aligned} g_{\mu\nu}(x, y) &= q_{\mu\nu} - (\mathring{E}_{\mu\nu} + A_{\mu\nu})y^2 - \frac{2}{l}(\mathring{E}_{\mu\nu} + A_{\mu\nu})y^3 + \frac{1}{12}\left(\left(\square\mathring{E}_{\mu\nu} - \frac{32}{l^2}\mathring{E}_{\mu\nu} + 2\mathring{R}_{\mu\alpha\nu\beta}\mathring{E}^{\alpha\beta} + 6\mathring{E}_\mu^\alpha\mathring{E}_{\alpha\nu}\right)\right. \\ &\quad \left.+\left(\square A_{\mu\nu} - \frac{32}{l^2}A_{\mu\nu} + 2(\nabla_{[\nu}K_{\mu\alpha\beta]})A^{\alpha\beta} + 2K_{\mu\gamma}^\gamma K_{\alpha\beta}^\beta A^{\alpha\beta} + 6\mathring{A}_\mu^\alpha\mathring{A}_{\alpha\nu}\right)\right)y^4 + \dots \end{aligned}$$

where

$$\begin{aligned} A_{\mu\nu} &= \left(\nabla_{[\delta}K_{\alpha\beta]}^\kappa + K_{\gamma[\beta}^\kappa K_{\alpha\delta]}^\gamma\right)n_\kappa n^\delta q_\mu^\alpha q_\nu^\beta + \frac{1}{6}q_{\mu\nu}\left(2\nabla^\lambda K_{\lambda\tau}^\tau - K_{\tau\lambda}^\lambda K_{\tau\gamma}^\gamma + K_{\tau\gamma\lambda} K_{\tau\lambda\gamma}^\gamma\right) \\ &\quad - \frac{2}{3}(q_\mu^\alpha q_\nu^\beta + n^\alpha n^\beta q_{\mu\nu})\left(\nabla_{[\lambda}K_{\beta\alpha]}^\lambda + K_{\gamma\lambda}^\lambda K_{\beta\alpha}^\gamma - K_{\beta\gamma}^\sigma K_{\sigma\alpha}^\gamma\right) \end{aligned}$$

and \square denotes the usual d’Alembertian. As in [4], it shows in particular that the propagating effect of 5D gravity arises only at the fourth order of the expansion. For a static spherical metric on the brane given by

$$g_{\mu\nu}dx^\mu dx^\nu = -F(r)dt^2 + \frac{dr^2}{H(r)} + r^2 d\Omega^2, \quad (7)$$

the projected Weyl term on the brane is given by the expressions¹

$$\begin{aligned}
E_{00} &= \frac{F}{r} \left(H' - \frac{1-H}{r} \right) + \left(\nabla_\nu K^{\mu 00} - \nabla_0 K^\mu_{0\nu} + K^\mu_{\gamma\nu} K^\gamma_{00} - K^\mu_{\gamma 0} K^\gamma_{0\nu} \right) n_\mu n^\nu F^2 \\
&\quad - \frac{2}{3} F(F-1) \left(\nabla_\lambda K^\lambda_{00} - \nabla_0 K^\lambda_{0\lambda} + K^\lambda_{\gamma\lambda} K^\gamma_{00} - K^\sigma_{0\gamma} K^\gamma_{\sigma 0} \right) + \frac{1}{6} F \left(2\nabla^\lambda K^\tau_{\lambda\tau} - K_{\tau\lambda}^\lambda K^{\tau\gamma} + K_{\tau\gamma\lambda} K^{\tau\lambda\gamma} \right), \\
E_{rr} &= -\frac{1}{rH} \left(\frac{F'}{F} - \frac{1-H}{r} \right) + \left(\nabla_\nu K^{\mu rr} - \nabla_r K^\mu_{r\nu} + K^\mu_{\gamma\nu} K^\gamma_{rr} - K^\mu_{\gamma r} K^\gamma_{r\nu} \right) n^\mu n^\nu H^{-2} \\
&\quad - \frac{2}{3} H^{-1} (H^{-1} - (n^r)^2) \left(\nabla_\lambda K^\lambda_{rr} - \nabla_r K^\lambda_{r\lambda} + K^\lambda_{\gamma\lambda} K^\gamma_{rr} - K^\sigma_{r\gamma} K^\gamma_{\sigma r} \right) \\
&\quad + \frac{1}{6} H^{-1} \left(2\nabla^\lambda K^\tau_{\lambda\tau} - K_{\tau\lambda}^\lambda K^{\tau\gamma} + K_{\tau\gamma\lambda} K^{\tau\lambda\gamma} \right), \\
E_{\theta\theta} &= -1 + H + \frac{r}{2} H \left(\frac{F'}{F} + \frac{H'}{H} \right) + \left(\nabla_\nu K^{\mu\theta\theta} - \nabla_\theta K^\mu_{\theta\nu} + K^\mu_{\gamma\nu} K^\gamma_{\theta\theta} - K^\mu_{\gamma\theta} K^\gamma_{\theta\nu} \right) n_\mu n^\nu r^4 \\
&\quad - \frac{2}{3} r^2 (r^2 + 1) \left(\nabla_\lambda K^\lambda_{\theta\theta} - \nabla_\theta K^\lambda_{\theta\lambda} + K^\lambda_{\gamma\lambda} K^\gamma_{\theta\theta} - K^\sigma_{\theta\gamma} K^\gamma_{\sigma\theta} - \frac{1}{2} \nabla^\lambda K^\tau_{\lambda\tau} + \frac{1}{4} K_{\tau\lambda}^\lambda K^{\tau\gamma} - \frac{1}{4} K_{\tau\gamma\lambda} K^{\tau\lambda\gamma} \right). \quad (8)
\end{aligned}$$

Note that in Eq.(7) the metric reduces to the Schwarzschild one, if $F(r)$ equals $H(r)$. The exact determination of these radial functions remains an open problem in black hole theory on the brane [4].

These components allow one to evaluate the metric coefficients in Eq.(7). The area of the 5D horizon is determined by $g_{\theta\theta}$. Defining $\psi(r)$ as the deviation from a Schwarzschild form H , i.e.,

$$H(r) = 1 - \frac{2M}{r} + \psi(r), \quad (9)$$

where M is constant, yields

$$\begin{aligned}
g_{\theta\theta}(r, y) &= r^2 + \psi' \left(1 + \frac{2}{l} y \right) + \left(\nabla_\nu K^{\mu\theta\theta} - \nabla_\theta K^\mu_{\theta\nu} + K^\mu_{\gamma\nu} K^\gamma_{\theta\theta} - K^\mu_{\gamma\theta} K^\gamma_{\theta\nu} \right) n_\mu n^\nu r^4 \\
&\quad - \frac{2}{3} r^2 (r^2 + 1) \left(\nabla_{[\lambda} K^\lambda_{\theta\theta]} + K^\lambda_{\gamma\lambda} K^\gamma_{\theta\theta} - K^\sigma_{\theta\gamma} K^\gamma_{\sigma\theta} - \frac{1}{2} \nabla^\lambda K^\tau_{\lambda\tau} + \frac{1}{4} K_{\tau\lambda}^\lambda K^{\tau\gamma} - \frac{1}{4} K_{\tau\gamma\lambda} K^{\tau\lambda\gamma} \right) y^2 \\
&\quad + \dots
\end{aligned} \quad (10)$$

It shows how ψ and the contortion and its derivatives determine the variation in the area of the horizon along the extra dimension. Also, the variation in the black string properties can be extracted. Obviously, when the torsion goes to zero, all the results above are led to the ones obtained in [4], [8], and references therein. In particular, Eq.(7) — when the torsion, and consequently $A_{\mu\nu}$ defined in Eq.(7), goes to zero — is led to the results previously obtained in [4].

As the area of the a black hole 5D horizon is determined by $g_{\theta\theta}$, in particular it may indicate observable signatures of corrections induced by contortion terms, since for a given fixed effective extra dimension size, supermassive black holes give the upper limit of variation in luminosity of quasars. Also, it is possible to re-analyze how the quasar luminosity variation behaves as a function of the AdS₅ bulk radius — corrected by contortion terms — in some solar mass range, as in [5] and references therein.

Furthermore, braneworld measurable corrections induced by contortion terms for quasars, associated with Schwarzschild and Kerr black holes, by their luminosity observation are important. These corrections in a torsionless context were shown to be more notorious for mini-black holes, where the Reissner-Nordstrom radius in a braneworld scenario is shown to be around a hundred times bigger than the standard ReissnerNordstrom radius associated with mini-black holes, besides mini-black holes being much more sensitive to braneworld effects. It is possible to repeat all the comprehensive and computational procedure in [5] in order to verify how the contortion effects in Eq.(10) can modify even more the above-mentioned results. In addition, such corrections involving torsion can also affect the mini-black holes radii (horizons), considering braneworld effects in ADD and Randall-Sundrum models, as presented in [5], where the radius of a black holes on the brane is much smaller than the size scale of the extra dimensions, and the black hole can be well described by the classical solutions of higher-dimensional Einstein equations. The

¹ In the three expressions below, the indices r and θ strictly denote the coordinates, and can not be confounded with summation indices.

Schwarzschild radius, in the context of Myers-Perry extra-dimensional formalism, is shown to be significantly increased by tidal charge and spinning effects and can be also corrected by contortion terms, as presented in Eq.(10). Such corrections may give more precise calculations concerning cross sections, Planck and mini-black holes masses, and Hawking temperature, contributing in this way to a more complete, precise, and realistic analysis of mini-black holes production in the next generation of particle colliders, such as LHC. For details in an torsionless scenario see [5] and references therein.

Finally, the applications of Eq.(10) are not the main aim here, and shall be addressed in a forthcoming publication, although such applications evince the importance of Eq.(10) in the context where any physical quantity involving black string horizon can be deeper investigated in the light of contortion corrections.

The modification in the area of the black hole horizon due to torsion terms, whose functional form is depicted in Eq. (10), can be better appreciated in a specific basis, i. e., an explicit *ansatz* for the spacetime metric. This is, however, out of the scope of the present work. The important point here is that torsion terms do affect the black hole horizon and the departure from the usual (torsionless) case is precisely given by Eq. (10). In the next Section we extend and apply the braneworld sum rules to the case with torsion, considering some estimates of the torsion effects.

III. SUM RULES WITH TORSION

In this Section we shall derive the consistency conditions for braneworld scenarios embedded in a Riemann-Cartan manifold. The general procedure is quite similar to the one found in [9, 10] and we shall comprise some of the general formulation here, for the sake of completeness.

We start in a very general setup, analyzing a D -dimensional bulk spacetime geometry, endowed with a non-factorizable metric

$$\begin{aligned} ds^2 &= G_{AB}dX^A dX^B \\ &= W^2(r)g_{\alpha\beta}dx^\alpha dx^\beta + g_{ab}(r)dr^a dr^b, \end{aligned} \quad (11)$$

where $W^2(r)$ is the warp factor, X^A denotes the coordinates of the full D -dimensional bulk, x^α stands for the $(p+1)$ non-compact spacetime coordinates, and r^a labels the $(D-p-1)$ directions in the internal compact space. The D -dimensional Ricci tensor can be related to its lower dimensional partners by [9]

$$R_{\mu\nu} = \bar{R}_{\mu\nu} - \frac{g_{\mu\nu}}{(p+1)W^{p-1}}\nabla^2 W^{p+1}, \quad (12)$$

$$R_{ab} = \tilde{R}_{ab} - \frac{p+1}{W}\nabla_a \nabla_b W, \quad (13)$$

where \tilde{R}_{ab} , ∇_a and ∇^2 are respectively the Ricci tensor, the covariant derivative, and the Laplacian operator constructed by means of the internal space metric g_{ab} . $\bar{R}_{\mu\nu}$ is the Ricci tensor derived from $g_{\mu\nu}$. Denoting the three curvature scalars by $R = G^{AB}R_{AB}$, $\bar{R} = g^{\mu\nu}\bar{R}_{\mu\nu}$, and $\tilde{R} = g^{ab}\tilde{R}_{ab}$ we have, from Eqs.(11) and (13),

$$\frac{1}{p+1}\left(W^{-2}\bar{R} - R^\mu_\mu\right) = pW^{-2}\nabla W \cdot \nabla W + W^{-1}\nabla^2 W \quad (14)$$

and

$$\frac{1}{p+1}\left(\tilde{R} - R^a_a\right) = W^{-1}\nabla^2 W, \quad (15)$$

where $R^\mu_\mu \equiv W^{-2}g^{\mu\nu}R_{\mu\nu}$ and $R^a_a \equiv g^{ab}R_{ab}$ ($R = R^\mu_\mu + R^a_a$). It can be easily verified that for an arbitrary constant ξ the following identity holds

$$\frac{\nabla \cdot (W^\xi \nabla W)}{W^{\xi+1}} = \xi W^{-2}\nabla W \cdot \nabla W + W^{-1}\nabla^2 W. \quad (16)$$

Combining the above equation with Eqs.(11) and (13) we have

$$\nabla \cdot (W^\xi \nabla W) = \frac{W^{\xi+1}}{p(p+1)}[\xi(W^{-2}\bar{R} - R^\mu_\mu) + (p-\xi)(\tilde{R} - R^a_a)]. \quad (17)$$

The D -dimensional Einstein equation is given by

$$R_{AB} = 8\pi G_D \left(T_{AB} - \frac{1}{D-2}G_{AB}T \right), \quad (18)$$

where G_D is the gravitational constant in D dimensions. It is easy to write down the following equations:

$$R^\mu_\mu = \frac{8\pi G_D}{D-2}(T^\mu_\mu(D-p-3) - T^m_m(p+1)), \quad R^m_m = \frac{8\pi G_D}{D-2}(T^m_m(p-1) - T^\mu_\mu(D-p-1)). \quad (19)$$

In the above equations we set $T^\mu_\mu = W^{-2}g_{\mu\nu}T^{\mu\nu}$ ($T_M^M = T^\mu_\mu + T^m_m$). Now, it is possible to relate R^μ_μ and R^m_m in Eq.(17) in terms of the stress-tensor. Note that the left hand side of Eq.(17) vanishes upon integration along a compact internal space. Hence, taking all that into account we have

$$\oint W^{\xi+1} \left(T^\mu_\mu [(p-2\xi)(D-p-1) + 2\xi] + T^m_m p (2\xi - p + 1) + \frac{D-2}{8\pi G_D} [(p-\xi)\tilde{R} + \xi\bar{R}W^{-2}] \right) = 0. \quad (20)$$

Heretofore in this Section we have just reproduced the well established results on consistency conditions applied to braneworld scenarios. Going further, the last equation — which provides a one parameter family of consistency conditions for warped braneworld scenarios in arbitrary dimensions — is applied to the case when torsion terms are present on the brane. First, however, we want to particularize the analysis for a 5-dimensional bulk, since it describes the phenomenologically interesting case. Besides, it makes the conclusions obtained here applicable to the case studied in [6], in continuity to the program of developing formal concepts to braneworld scenarios with torsion. In this way $D = 5$, $p = 3$, and consequently $\tilde{R} = 0$, because there is just one dimension on the internal space. With such specifications and assuming implicitly, as usual, that the brane action volume element does not depend on torsion², Eq.(20) becomes

$$\oint W^{\xi+1} \left(T^\mu_\mu + 2(\xi-1)T^m_m + \frac{\xi}{\kappa_5^2}\bar{R}W^{-2} \right) = 0, \quad (21)$$

where $8\pi G_5 = \kappa_5^2 = \frac{8\pi}{M_5^3}$, with M_5 denoting the 5-dimensional Planck mass. In order to implement torsion terms in our analysis, the expressions for the Riemann and Ricci tensors in terms of contortion components related with their partners — constructed with the usual metric compatible Levi-Civita connection [11]

$$\bar{R}^\lambda_{\tau\alpha\beta} = \dot{\bar{R}}^\lambda_{\tau\alpha\beta} + \nabla_{[\alpha} K^\lambda_{\tau\beta]} + K^\lambda_{\gamma[\alpha} K^\gamma_{\tau\beta]}, \quad \bar{R}_{\tau\beta} = \dot{\bar{R}}_{\tau\beta} + \nabla_{[\lambda} K^\lambda_{\tau\beta]} + K^\lambda_{\gamma\lambda} K^\gamma_{\tau\beta} - K^\lambda_{\tau\gamma} K^\gamma_{\lambda\beta} \quad (22)$$

are used, where ∇ precisely denotes the covariant derivative without torsion. The brane scalar curvature is written in terms of the usual scalar curvature, $\dot{\bar{R}}$ (without torsion), plus 4-dimensional contortion terms corrections [11]

$$\bar{R} = \dot{\bar{R}} + 2D^\lambda {}^{(4)}K^\tau_{\lambda\tau} - {}^{(4)}K^\lambda_{\tau\lambda} {}^{(4)}K^\tau_{\lambda} + {}^{(4)}K_{\tau\gamma\lambda} {}^{(4)}K^{\tau\lambda\gamma}, \quad (23)$$

where the label “(4)” on the contortion terms denotes the contortion of the 3-branes and the covariant derivative is considered when a connection that presents *no* torsion is taken into account. In order to verify the influence of torsion terms in the scalar curvature within the braneworld context we shall substitute Eq.(23) into (21). First, however, note that in order to reproduce the observable universe one can put $\dot{\bar{R}} = 0$ with $10^{-120} M_{Pl}$ of confidence level, where M_{Pl} is the Planck mass. Note that the observations concerning the scalar curvature are related to the torsionless $\dot{\bar{R}}$, not to \bar{R} . So, taking it into account it follows that

$$\oint W^{\xi+1} \left[T^\mu_\mu + 2(\xi-1)T^m_m + \frac{\xi W^{-2}}{\kappa_5^2} \left(2D^\lambda {}^{(4)}K^\tau_{\lambda\tau} - {}^{(4)}K^\lambda_{\tau\lambda} {}^{(4)}K^\tau_{\lambda} + {}^{(4)}K_{\tau\gamma\lambda} {}^{(4)}K^{\tau\lambda\gamma} \right) \right] = 0. \quad (24)$$

In order to proceed with the consistency conditions we specify the standard *ansatz* for the stress-tensor. Assuming that there are no other types of matter in the bulk, except the branes and the cosmological constant, we have [10]

$$T_{MN} = -\frac{\Lambda}{\kappa_5^2}G_{MN} - \sum_i T_3^{(i)} P[G_{MN}]_3^{(i)} \delta(y - y_i), \quad (25)$$

² In this way, we guarantee that the brane volume element reduces to d^4x in the limit of null torsion and flat space.

where Λ is the bulk cosmological constant, $T_3^{(i)}$ is the tension associated to the i^{th} -brane and $P[G_{MN}]_3^{(i)}$ is the pull-back of the metric to the 3-brane. The partial traces of (25) are given by

$$T_\mu^\mu = \frac{-4\Lambda}{\kappa_5^2} - 4 \sum_i T_3^{(i)} \delta(y - y_i), \quad \text{and} \quad T_m^m = -\frac{\Lambda}{\kappa_5^2}, \quad (26)$$

in such way that Eq.(24) becomes

$$\oint W^{\xi+1} \left[\frac{2\Lambda}{\kappa_5^2} (\xi + 1) + 4 \sum_i T_3^{(i)} \delta(y - y_i) - \frac{\xi W^{-2}}{\kappa_5^2} \left(2D^\lambda {}^{(4)}K_{\lambda\tau}^\tau - {}^{(4)}K_{\tau\lambda}^\lambda {}^{(4)}K_{\lambda}^{\tau\lambda} + {}^{(4)}K_{\tau\gamma\lambda} {}^{(4)}K^{\tau\lambda\gamma} \right) \right] = 0. \quad (27)$$

As one can see, this formalism can be applied for a several branes scenario. The number of branes, nevertheless, is not so important to our analysis. To fix ideas let us particularize the formalism to the two branes case. Denoting $T_3^{(1)} = \lambda$, the visible brane, $T_3^{(2)} = \tilde{\lambda}$, and assuming that neither the cosmological constant nor the branes contortion terms do depend on the extra dimension, Eq.(27) gives

$$4\lambda W_\lambda^{\xi+1} + 4\tilde{\lambda} W_{\tilde{\lambda}}^{\xi+1} + \frac{2\Lambda}{\kappa_5^2} (\xi + 1) \oint W^{\xi+1} - \frac{\xi}{\kappa_5^2} \left(2D^\lambda {}^{(4)}K_{\lambda\tau}^\tau - {}^{(4)}K_{\tau\lambda}^\lambda {}^{(4)}K_{\lambda}^{\tau\lambda} + {}^{(4)}K_{\tau\gamma\lambda} {}^{(4)}K^{\tau\lambda\gamma} \right) \oint W^{\xi-1} = 0, \quad (28)$$

where $W_\lambda = W(y = y_1)$ and $W_{\tilde{\lambda}} = W(y = y_2)$. Now some physical outputs of the general Eq.(28) are analyzed, in order to investigate the viability of braneworld scenarios with torsion. The first case we shall look at relates a factorizable geometry. Nevertheless, before going forward, we shall emphasize that if one implements the torsion null case in Eq.(28), it is easy to see that for $\xi = -1$ one recovers the well known fine tuning of the Randall-Sundrum model, i.e.,

$$\lambda + \tilde{\lambda} = 0, \quad (29)$$

as expected.

A. Non-warped compactifications with torsion

The non-warped case is implemented by imposing $W = 1$, working then in a factorizable spacetime geometry. The general approach on consistency conditions, as exposed before, allows this possibility. In this Subsection we are therefore concerned with the viability of braneworld scenarios in the general scope analyzed in reference [12] and in the presence of torsion. The case we are going to describe here is not the most interesting. We shall, however, study a little further this simplified case, since it can provide some physical insight to the warped case.

From Eq.(28), the non-warped case reads

$$\frac{2\Lambda}{\kappa_5^2} (\xi + 1)V + 4\lambda + 4\tilde{\lambda} - \frac{\xi V}{\kappa_5^2} \left(2D^\lambda {}^{(4)}K_{\lambda\tau}^\tau - {}^{(4)}K_{\tau\lambda}^\lambda {}^{(4)}K_{\lambda}^{\tau\lambda} + {}^{(4)}K_{\tau\gamma\lambda} {}^{(4)}K^{\tau\lambda\gamma} \right) = 0, \quad (30)$$

where V denotes the “volume” of the internal space. Note that for $\xi = 0$, the torsion terms do not influence the general sum rules in the present case. In fact, for $\xi = 0$ it follows that

$$\frac{V\Lambda}{2\kappa_5^2} + \lambda + \tilde{\lambda} = 0, \quad (31)$$

which states that, for non-warped branes, it is possible to exist an AdS_5 bulk, even for strictly positive tension values associated with the branes. Another interesting case is obtained for $\xi = -1$. In such case the bulk cosmological constant is factored out and consequently

$$\left({}^{(4)}K_{\tau\lambda}^\lambda {}^{(4)}K^{\tau\lambda} - {}^{(4)}K_{\tau\gamma\lambda} {}^{(4)}K^{\tau\lambda\gamma} - 2D^\lambda {}^{(4)}K_{\lambda\tau}^\tau \right) = \frac{4\kappa_5^2}{V} (\lambda + \tilde{\lambda}). \quad (32)$$

Note that the left hand side (LHS) of (32) can be interpreted, from Eq.(23), as the difference between $\overset{\circ}{R}$ and \bar{R} . In other words, the LHS of Eq.(32) measures the contribution of the torsion terms to the brane curvature, i. e., it indicates how much the brane curvature differs itself from zero, due to torsion terms. So, we can write schematically

$$\overset{\circ}{R} - \bar{R} = \frac{4\kappa_5^2}{V} (\lambda + \tilde{\lambda}). \quad (33)$$

We see that the effect of the torsion in the brane curvature is proportional to the branes tension values in the two branes scenario, but it decreases with the distance between the branes. Moreover, since $\kappa_5^2 = 8\pi G_5 \sim 1/M_5^3$, such an effect is about $1/(VM_5^3)$. Therefore, it indicates the low magnitude of torsion effects in the braneworld scenario with large extra transverse dimension, since it is suppressed by the 5-dimensional Planck scale and also by the volume of the internal space. Obviously, in a braneworld scenario which solves the hierarchy problem the typical scale of the higher dimensional Planck mass is of order $M_5 \sim M_{weak}$ and then, the suppression due to the internal space volume is attenuated.

B. The warped case

This is the most general case we consider in this Section. In the absence of a factorizable geometry, some configurations of the warp factor may be responsible for the right mass partition in the Higgs mechanism without the necessity of any additional hierarchy [1]. Starting from the general Eq.(28), we shall look at the most important cases, namely $\xi = -1, 0, 1$.

For $\xi = -1$ we have

$$\left({}^{(4)}K_{\tau\lambda}{}^\lambda {}^{(4)}K^{\tau\lambda} - {}^{(4)}K_{\tau\gamma\lambda} {}^{(4)}K^{\tau\lambda\gamma} - 2D^\lambda {}^{(4)}K_{\lambda\tau}^\tau \right) = \frac{4\kappa_5^2}{\oint W^{-2}}(\lambda + \tilde{\lambda}). \quad (34)$$

This is the warped analogue of Eq.(32) with the volume of the internal space replaced by the circular integral of W^{-2} in the denominator of the right hand side. The same conclusions as the $\xi = -1$ case of the previous Subsection still hold, but here we call the attention to the minuteness of the torsion terms: even contributing with such low magnitude effect to the brane curvature, it allows the branes to have both the same sign associated to their respective tension values.

The bulk spacetime type can be better visualized in the $\xi = 0$ case. Since all torsion terms of Eq.(28) are factored out, it follows that

$$\frac{\Lambda}{2\kappa_5^2} \oint W + \lambda W_\lambda + \tilde{\lambda} W_{\tilde{\lambda}} = 0. \quad (35)$$

Therefore, as $\oint W < 0$, it is easy to see that if $\lambda, \tilde{\lambda} > 0$ then necessarily $\Lambda > 0$ corresponding to an dS_5 bulk geometry. Otherwise, being $\lambda, \tilde{\lambda} < 0$ one arrives at an AdS_5 bulk geometry.

For the $\xi = 1$ case, a slight modification of Eq.(34) deserves a notification. The implementation of $\xi = 1$ in the Eq.(28) results in

$$\frac{\Lambda}{\kappa_5^2} \oint W^2 + \lambda W_\lambda^2 + \tilde{\lambda} W_{\tilde{\lambda}}^2 - \frac{V}{4\kappa_5^2} \left(2D^\lambda {}^{(4)}K_{\lambda\tau}^\tau - {}^{(4)}K_{\tau\lambda}{}^\lambda {}^{(4)}K^{\tau\lambda} + {}^{(4)}K_{\tau\gamma\lambda} {}^{(4)}K^{\tau\lambda\gamma} \right) = 0. \quad (36)$$

Now, isolating the torsion contribution to the curvature we have

$$\left(2D^\lambda {}^{(4)}K_{\lambda\tau}^\tau - {}^{(4)}K_{\tau\lambda}{}^\lambda {}^{(4)}K^{\tau\lambda} + {}^{(4)}K_{\tau\gamma\lambda} {}^{(4)}K^{\tau\lambda\gamma} \right) = \frac{4\Lambda}{V} \oint W^2 + \frac{4\kappa_5^2}{V} (\lambda W_\lambda^2 + \tilde{\lambda} W_{\tilde{\lambda}}^2). \quad (37)$$

From Eq.(37) we see that the torsion contribution to the brane curvature is constrained by the internal space volume, however terms coming from the warped compactification — as $\oint W^2$ and $\oint W_{\lambda,\tilde{\lambda}}^2$ — can turn this contribution more appreciable. In particular, the first term of the right hand side of (37) is the dominant one, since it is not suppressed by the 5-dimensional Planck scale and it is multiplied by the bulk cosmological constant. We shall make more comments about these results in the next Section.

IV. CONCLUDING REMARKS AND OUTLOOK

This paper concerns some effects evinced by torsion terms corrections — both in the bulk and on the brane. To study a typical gravitational signature arising from a gravitational system we performed in Section II the analysis based upon the well known Taylor expansion tools — strongly reminiscent of the assumption of a direction orthogonal to the brane — of the bulk metric in terms of the brane metric, taking into account bulk torsion terms. Our main result is summarized by Eq.(10). It shows how the bulk torsion terms intervene in the black hole area, in an attempt to find some observable effects arising from the torsion properties. Again, its highly non-trivial form can be better

studied in the context of a specific model. It is out of the scope of this paper, nevertheless we shall point a line of research in this area. It could be interesting to apply the results found in this Section to some gravitational systems, in analogy to what was accomplished in standard braneworld scenarios (see, for instance, references [5]).

In order to study the behavior of the brane torsion terms we extend, in Section III, the braneworld sum rules. It was demonstrated that the consistency conditions do not preclude the possibility of torsion on the brane. It was shown, however, that the torsion effects in the brane curvature are suppressed. It could, in principle, explain a negative result for experiments from the geometrical point of view. Just for a comparative complement, in reference [3] the 5-dimensional torsion field, identified with the rank-2 Kalb-Ramond (KR) field, was considered in the bulk. It was demonstrated by the authors the existence of an additional exponential damping for the zero-mode of the KR field arisen from the compactification of the transverse dimension. In some sense, our purely geometrical sum rules complete the analysis concerning the presence of torsion, this time on the brane.

In this paragraph we would like to call attention for some related issues appearing in the literature. In [3, 13] it was shown that in an effective 4-dimensional theory on the visible brane, the KR field — as a source of torsion — is suppressed when a torsion-dilaton-gravity action in a Randall-Sundrum braneworld scenario is considered, explaining the apparent insensitivity of torsion in the brane. It was shown, however, that even in this case the KR field may lead to new signatures in TeV scale experiments, when a coupling between dilaton and torsion is taken into account. The warped extra-dimensional formalism points to the presence of new interactions, of significant phenomenological importance, between the Kaluza-Klein modes of the dilaton and the KR field.

Briefly speaking, the results of this paper point to the fact that the hypothesis of a torsionless brane universe may be based upon a justified impression, since its effects from the bulk (studied from a quantum field theory approach) and from the brane (analyzed via the geometrical sum rules) are suppressed by some damping factor. We emphasize, however, that in the context above, the naive estimative of the 4-dimensional torsion effects (37) must be complemented by the results of a more specific system, perhaps coming from the Section II results, via Eq. (10). Such a characterization may put this gravitational and geometrical approach in the same level, concerning the brane torsion phenomenology, as, for instance, the massive spectrum of 5-dimensional KR field signature which can be viewed in a TeV-scale accelerator [3]. In this vein, the torsionless brane universe may be naturally substituted by a more fidedigne braneworld scenario that contains torsion, and may be useful to a more precise description of physical theories.

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